TRINITY COLLEGE

YEAR 12
MATHEMATICS
SPECIALIST

Test 3, 2023
Section One: Calculator Free
Integration Techniques and Differential Equations

STUDENT'S NAME: Solutions [LAWRENCE]

DATE: Wednesday 16th August

TIME: 25 minutes

MARKS: 25
ASSESSMENT %: 10

INSTRUCTIONS:
Standard Items: Pens, pencils, drawing templates, eraser
Special Items:

Questions or parts of questions worth more than 2 marks require working to be shown to receive full marks.

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Question 1 **(8 marks)**

(a) $\int x^2\sqrt{3+x^3}\,dx$ (2 marks)

let $u = 3 + x^3$

$\frac{du}{dx} = 3x^2$

$dx = \frac{du}{3x^2}$

$= \int \cancel{x^2}\sqrt{u}\,\frac{du}{3\cancel{x^2}}$

$= \frac{1}{3}\int u^{1/2}\,du$

$= \frac{1}{3}\left(\frac{2}{3}u^{3/2}\right) + c$

$= \frac{2}{9}(3+x^3)^{3/2} + c$

✓ identifying chain
✓ scale factor and +c

(b) $\int x\sqrt{x+1}\,dx$ Use the substitution $u = x + 1$ (3 marks)

let $u = x + 1$

$x = u - 1$

$\frac{du}{dx} = 1$

$du = dx$

$= \int (u-1)\,u^{1/2}\,du$

$= \int u^{3/2} - u^{1/2}\,du$

$= \frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2} + c$

$= \frac{2}{5}(x+1)^{5/2} - \frac{2}{3}(x+1)^{3/2} + c$

✓ substitution of u & x

✓ new integral in terms of u

✓ integrates both terms correctly

(c) $\int \sin^2 x \cos^3 x \; dx$ (3 marks)

$\cos^3 x = \cos^2 x \cdot \cos x$

$\cos^2 x = 1 - \sin^2 x$

$= \int \sin^2 x \; \cos^2 x \; \cos x \, dx$

$= \int \sin^2 x \, (1 - \sin^2 x) \cos x \; dx$

$= \int \sin^2 x \cos x \; - \; \sin^4 x \; \cos x \; dx$

$= \frac{1}{3} \sin^3 x \; - \; \frac{1}{5} \sin^5 x \; + \; c$

✓ substituting $\cos^3 x$ with $\cos^2 x \cos x$ & $\cos^2 x$ with $(1 - \sin^2 x)$

✓ new integral

✓ integrates both terms correctly.

Question 2 **(5 marks)**

(a) Express $\frac{2x+1}{x^2(x+1)}$ in the form $\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x+1}$ (3 marks)

$2x+1 = A(x)(x+1) + B(x+1) + C(x^2)$

$2x+1 = Ax^2 + Ax + Bx + B + Cx^2$

$2x+1 = (A+C)x^2 + (A+B)x + B$

$\therefore B = 1$ $A + B = 2$ $A + C = 0$

$A = 1$ $C = -1$

$\frac{1}{x} + \frac{1}{x^2} - \frac{1}{x+1}$

✓ Multiplying A, B & C by correct factors

✓ equating coefficients & solving for A, B, C

✓ correct partial fraction.

(b) Hence, determine $\int \frac{2x+1}{x^2(x+1)}\,dx$ (2 marks)

$= \int \frac{1}{x} + \frac{1}{x^2} - \frac{1}{x+1}\,dx$

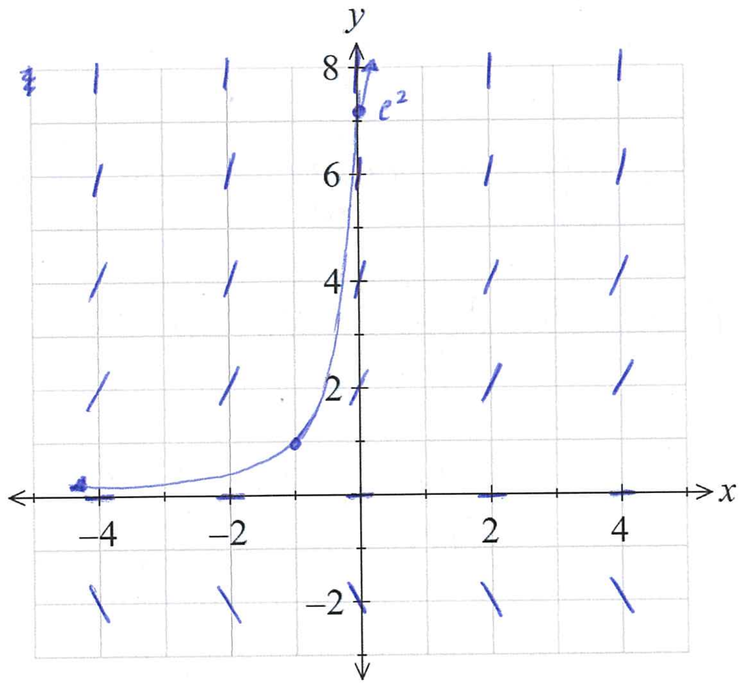
$= \ln|x| - \frac{1}{x} - \ln|x+1| + c$

✓ rewrites integral with partial fractions from a)

✓ integrates each term correctly & +c

Question 3 **(6 marks)**

(a) On the axis below, sketch the slope field for the differential equation $\frac{dy}{dx} = 2y$. (2 marks)



✓ shows 0 slopes

✓ shows some +ve & −ve slopes.

(b) If $y(-1) = 1$, solve the differential given in part (a) to find y in terms of x. (3 marks)

$(-1, 1)$

$\frac{dy}{dx} = 2y$

$\int \frac{1}{y}\, dy = \int 2\, dx$

$\ln|y| = 2x + c$

$e^{\ln y} = e^{2x + c}$

$y = e^{2x} . c_2$

$y = Ae^{2x}$

$(-1, 1) \quad 1 = Ae^{-2}$

$A = \frac{1}{e^{-2}}$

$A = e^2$

$\therefore y = e^2 . e^{2x}$

$y = e^{2x+2}$

✓ solves differential

✓ solves for A

✓ correct y equation

(c) Sketch the graph of the solution curve found in part (b) on the slope field in part (a). (1 mark)

✓ correct curve must go through (−1, 1) & be asymptotic

Question 4 **(5 marks)**

The curve $x^3 + y^3 - 9xy = 0$, known as a *folium*, dates back to Descartes in the 1630s.

(a) Determine $\dfrac{dy}{dx}$. (3 marks)

$$3x^2 + 3y^2\frac{dy}{dx} - 9x\frac{dy}{dx} - 9y = 0$$

$$(3y^2 - 9x)\frac{dy}{dx} = 9y - 3x^2$$

$$\frac{dy}{dx} = \frac{9y - 3x^2}{3y^2 - 9x}$$

$$= \frac{3y - x^2}{y^2 - 3x}$$

✓ Differentiates x^3 & y^3 correctly

✓ Differentiates $-9xy$ correctly

✓ Isolates $\dfrac{dy}{dx}$ (No need to simplify)

(b) Determine the equation of the tangent to the curve at the point (2, 4). (2 marks)

$$\left.\frac{dy}{dx}\right|_{(2,4)} = \frac{12 - 4}{16 - 6} = \frac{8}{10} = 0.8$$

$y = 0.8x + c$

$4 = 1.6 + c$

$c = 2.4$

✓ finds $\dfrac{dy}{dx}$ @ $x = 2$, $y = 4$

✓ correct c value

$\therefore y = 0.8x + 2.4$

$\left(y = \frac{4}{5}x + \frac{12}{5}\right)$

END OF QUESTIONS

TRINITY COLLEGE

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YEAR 12
MATHEMATICS
SPECIALIST

Test 3, 2023
Section Two: Calculator Allowed
Integration Techniques and Differential Equations

STUDENT'S NAME: ______________________________

DATE: Wednesday 16th August **TIME:** 25 minutes **MARKS:** 32
ASSESSMENT %: 10

INSTRUCTIONS:

Standard Items:	Pens, pencils, drawing templates, eraser
Special Items:	1 A4 page notes, Classpad, Scientific Calculator

Questions or parts of questions worth more than 2 marks require working to be shown to receive full marks.

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Question 5 **(4 marks)**

A refrigerator has a constant temperature of 3°C. A can of drink with temperature 30°C is placed in the refrigerator. After being in the refrigerator for 15 minutes, the temperature of the can of drink is 28°C. The change in the temperature of the can of drink can be modelled by $\frac{dT}{dt} = k(T-3)$, where T is the temperature of the can of drink, t is the time in minutes after the can is placed in the refrigerator and k is a constant.

(a) Show that $T = 3 + Ae^{kt}$, where A is a constant, satisfies $\frac{dT}{dt} = k(T-3)$. (1 mark)

$$\frac{dT}{dt} = k(T-3)$$

$$\int \frac{1}{T-3}\,dT = \int k\,dt$$

$$\ln|T-3| = kt + c$$

$$T-3 = e^{kt}\cdot e^{c}$$

$$T = Ae^{kt} + 3$$

✓ all steps for solving differential

(b) After 60 minutes, at what rate is the temperature of the can of drink changing? (3 marks)

$(0, 30)$

$(15, 28)$

$$30 = Ae^{0} + 3$$

$$A = 27$$

✓ uses $(0,30)$ to solve for A

$$T = 3 + 27e^{kt}$$

$$28 = 3 + 27e^{15k}$$

✓ uses $(15,28)$ to solve for k

use CAS

$$k \approx -0.0051307$$

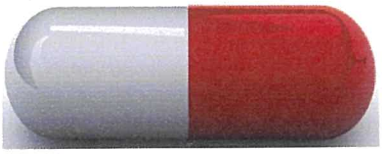
✓ substitutes in $t = 60$ to final T formula.

$$T = 3 + 27e^{-0.0051307}$$

$$\left.\frac{dT}{dt}\right|_{t=60} = -0.102\ °C/min$$

Question 6 **(5 marks)**

A balloon is in the shape of a cylinder and has hemispherical ends of the same radius as that of the cylinder. (i.e., it looks like a medicine capsule). The balloon is being inflated at the rate of 261π cubic centimetres per minute. At the instant that the radius of the cylinder is 3 cm, the volume of the balloon is 144π cubic centimetres and the radius of the cylinder is increasing at the rate of 2 centimetres per minute.



$\frac{dV}{dt} = 261\pi\ cm^3/min$

a) At this instant, what is the height of the cylinder? (2 marks)

$r = 3$

$V = 144\pi$

$\frac{dr}{dt} = 2cm/min$

$$V = \frac{4}{3}\pi r^3 + \pi r^2 h$$

$$144\pi = \frac{4}{3}\pi(27) + 9\pi h$$

$$h = 12\ cm$$

✓ correct formula for V

✓ solves for h.

b) At this instant, how fast is the height of the cylinder changing? $\frac{dh}{dt} = ?$ (3 marks)

when $r = 3$ & $h = 12$

& $\frac{dr}{dt} = 2\ cm/min$

$$V = \frac{4}{3}\pi r^3 + \pi r^2 h$$

diff w.r.t t

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt} + \pi r^2 \frac{dh}{dt} + h(2\pi r)\frac{dr}{dt}$$

$$261\cancel{\pi} = 36\cancel{\pi}(2) + 9\cancel{\pi}\left(\frac{dh}{dt}\right) + 72\cancel{\pi}(2)$$

$$261 = 72 + 9\frac{dh}{dt} + 144$$

$$45 = 9\frac{dh}{dt}$$

$$\frac{dh}{dt} = 5\ cm/min$$

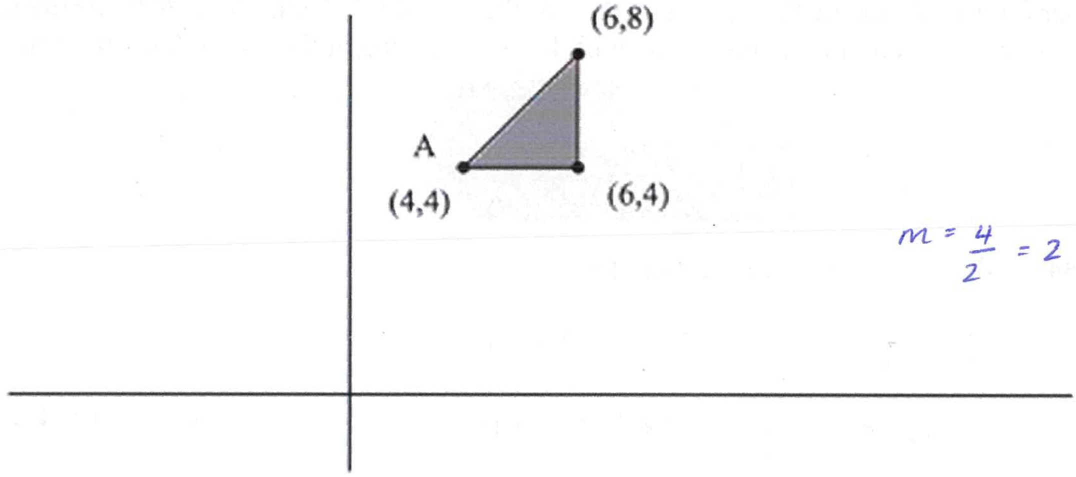
✓ Derives all terms wrt 't'

✓ substitutes in $r = 3$, $h = 12$, $\frac{dr}{dt} = 2$

✓ correct $\frac{dh}{dt}$

Question 7 **(10 marks)**

Consider the following shaded region.



$m = \frac{4}{2} = 2$

(a) Determine the volume of the object created by rotating the shaded region around the x axis. (2 marks)

$y = 2x + c$

$4 = 8 + c$

$c = -4$

$y = 2x - 4$

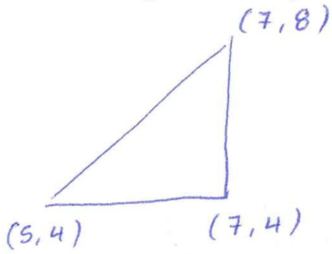
$V = \int_4^6 \pi\left[(2x-4)^2 - (4)^2\right] dx$

$= 134.04 \text{ units}^3$

✓ finds correct integral for V

✓ correct volume.

(b) Show that a horizontal translation of 1 unit to the right will not change the volume of the solid produced by rotating the shaded region around the x axis. (2 marks)



$y = 2(x-1) - 4$

$= 2x - 6$

$V = \int_5^7 \pi\left[(2x-6)^2 - (4)^2\right] dx$

$= 134.04 \text{ units}^3$

same as a)

✓ new integral for V

✓ correct volume.

(c) State the ~~horizontal~~ vertical translation which would need to take place which would give a volume of 500 units^2 when the shaded region is rotated around the x axis. (3 marks)

$$500 = \pi\int_4^6 (2x-4+a)^2 - (4+a)^2\,dx$$

$$= \pi\int_4^6 4x^2 - 16x + 4ax - 16a\,dx$$

$$500 = \pi\left[\frac{4}{3}x^3 - 8x^2 + 2ax - 16ax\right]_4^6$$

use CAS

$a = 14.561$

$\therefore$ vertical translation 14.561 units up.

✓ states correct integral to solve showing vertical translation of BOTH functions

✓ solves for a

✓ states translation

(d) Determine the volume of the object created by rotating the original shaded region around the y axis. How does this compare to your answer in part (a). (3 marks)

$y = 2x - 4$

$x = \dfrac{y+4}{2}$

$$V = \pi\int_4^8 (6)^2 - \left(\frac{y+4}{2}\right)^2 dy$$

$= 134.04$ units^3

same as part a)

✓ rearranges for $x =$

✓ finds new V

✓ makes statement

Question 8 **(13 marks)**

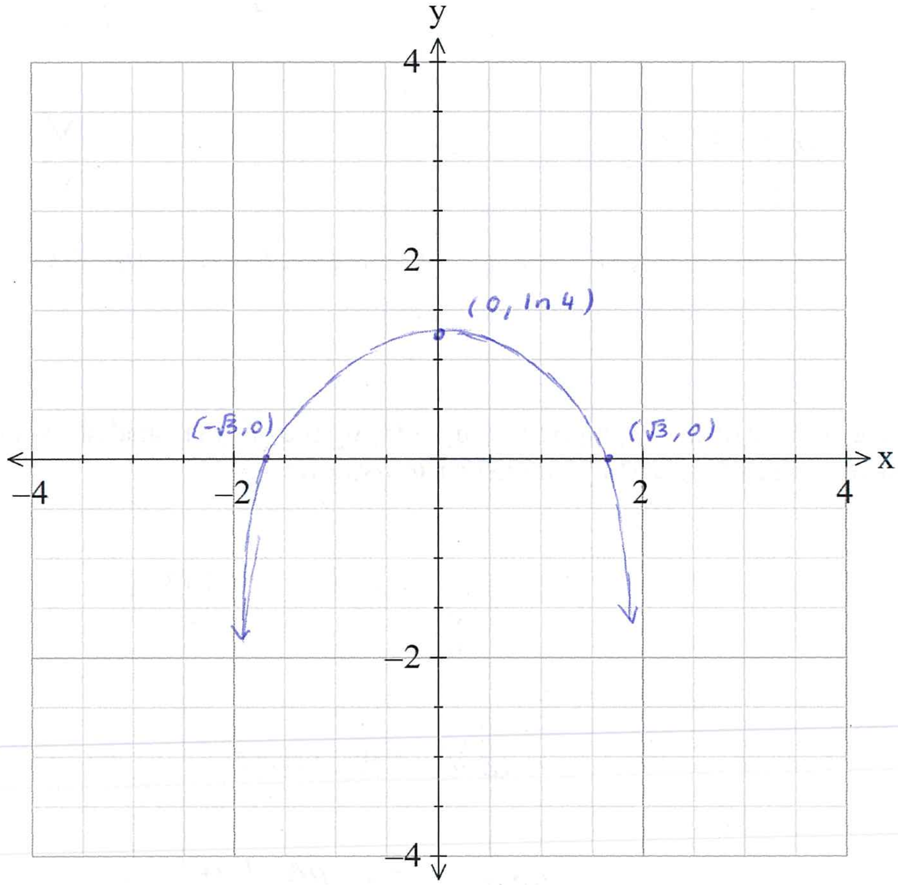
Consider the function $f(x) = \log_e(4 - x^2)$.

(a) Determine the largest possible domain for which $f(x)$ is defined. (1 mark)

$\{x : x \in \mathbb{R}, -2 < x < 2\}$

✓ correct domain

(b) Sketch the graph of $f(x)$, labelling all the key features and using exact values. (3 marks)



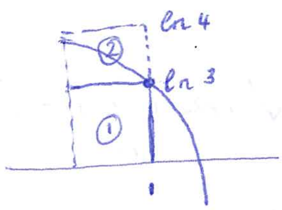
✓ shape

✓ correct x & y intercepts as exact values

✓ asymptotes at $x = \pm 2$

Let A be the magnitude of the area enclosed by the graph of $f(x)$, the co-ordinate axes and the line $x = 1$.

(c) Without evaluating A, use the graph of $f(x)$ to explain why $\log_e(3) < A < \log_e(4)$. (2 marks)



Area is ~~smaller~~ larger than rectangle 1

$L \times W$

$1 \times \ln 3$

$\ln 3$

Area is smaller than rectangle 2

$L \times W$

$1 \times \ln 4$

$\ln 4$

✓ discusses small rectangle area

✓ discusses large rectangle area

(d) i) Show that $\frac{d(x\log_e(4-x^2))}{dx} = \ln(4-x^2) - 2\cdot\frac{x^2}{4-x^2}$. (2 marks)
All steps of working must be shown.

$u = x \quad u' = 1$

$v = \ln(4-x^2) \quad v' = \frac{-2x}{4-x^2}$

$\frac{dy}{dx} = (1)(\ln(4-x^2)) + (x)\left(\frac{-2x}{4-x^2}\right)$

$= \ln(4-x^2) - \frac{2x^2}{4-x^2}$

✓ correct u' and v'

✓ correct use of product rule.

ii) Using the fact that $\frac{x^2}{4-x^2}$ can be written as $\frac{x^2-4+4}{4-x^2}$, show that

$\int \frac{x^2}{4-x^2}\,dx = \ln|x+2| - \ln[x-2] - x + c.$ (3 marks)

$(4-x^2) = (2-x)(2+x)$

$\int \frac{x^2-4+4}{4-x^2}\,dx = \int \frac{4}{4-x^2} - \frac{4-x^2}{4-x^2}\,dx$

$= \int \frac{4}{4-x^2} - 1\,dx$

$= \int \frac{4}{(2-x)(2+x)} - \int 1\,dx$

$= \int \frac{1}{2-x}\,dx + \int \frac{1}{2+x}\,dx - \int 1\,dx$

$= \ln|2-x| + \ln|2+x| - x + c$

$= \ln|2+x| - \ln|x-2| - x + c$

$\frac{4}{(2-x)(2+x)} = \frac{A}{2-x} + \frac{B}{2+x}$

$4 = 2A + Ax + B2 - Bx$

$A - B = 0$

$2A + 2B = 4$

$A = 1$

$B = 1$

✓ rearranges integral

✓ use of partial fractions to rearrange further

✓ integrates each term correctly

iii) Hence find the exact value of A in the form $a + b\log_e(c)$ where a, b and c are integers. (2 marks)

$\int_0^1 \ln(4-x^2) - 2\frac{x^2}{4-x^2}\,dx = x\ln(4-x^2)$

$\int_0^1 \ln(4-x^2)\,dx - 2\int_0^1 \frac{x^2}{4-x^2}\,dx = x\ln(4-x^2)$

$\int_0^1 \ln(4-x^2)\,dx = x\ln(4-x^2) + 2\int_0^1 \frac{x^2}{4-x^2}\,dx$

$= \left[x\ln(4-x^2) + 2(\ln|2+x| - \ln|x-2| - x\right]_0^1$

$= (\ln 3 + 2\ln 3 - 2\cancel{\ln 1} - 2) - (0 + 2\cancel{\ln 2} - 2\cancel{\ln 2} - 0)$

$= 3\ln 3 - 2$

$= -2 + 3\ln 3$

(can use CAS to solve)

✓ evidence of rearranging integral from a) and b)

✓ $a = -2, b = 3, c = 3$